ISRAEL JOURNAL OF MATHEMATICS, Vol. 39, No. 3, 1981

SPLITTING PROPERTIES AND JUMP CLASSES

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ABSTRACT

We show that the promptly simple sets of Maass form a filter in the lattice $\mathscr{E}$ of recursively enumerable sets. The degrees of the promptly simple sets form a filter in the upper semilattice of r.e. degrees. This filter nontrivially splits the high degrees ($\boldsymbol{a}$ is high if $\boldsymbol{a}' = \boldsymbol{0}''$). The property of prompt simplicity is neither definable in $\mathscr{E}$ nor invariant under automorphisms of $\mathscr{E}$. However, prompt simplicity is easily shown to imply a property of r.e. sets which is definable in $\mathscr{E}$ and which we have called the splitting property. The splitting property is used to answer many questions about automorphisms of $\mathscr{E}$. In particular, we construct low d-simple sets which are not automorphic, answering a question of Lerman and Soare. We produce classes invariant under automorphisms of $\mathscr{E}$ which nontrivially split the high degrees as well as all of the other classes of r.e. degrees defined in terms of the jump operator. This refutes a conjecture of Soare and answers a question of H. Friedman.

§0. Introduction

Let $\mathscr{E}$ denote the lattice of recursively enumerable sets and Aut($\mathscr{E}$) the group of automorphisms of $\mathscr{E}$. What properties of r.e. sets A and B guarantee that there is a $\Phi \in \mathrm{Aut}(\mathscr{E})$ such that $\Phi(A) = B$? The first nontrivial result on this question was by Soare [8, theorem 2.3] who showed that if A and B are maximal r.e. sets (A is maximal if the equivalence class of A is a coatom in $\mathscr{E}^* = \mathscr{E}$ modulo the ideal of finite sets), then there is a $\Phi \in \mathrm{Aut}(\mathscr{E})$ such that $\Phi(A) = B$. Let $\mathscr{L}^*(A)$ denote the principal filter generated by A in $\mathscr{E}^*$. Then Soare's theorem can be rephrased to say that if each of $\mathscr{L}^*(A)$ and $\mathscr{L}^*(B)$ is the two-element Boolean algebra, then A is automorphic to B. Clearly, if A is automorphic to B, $\mathscr{L}^*(A) \cong \mathscr{L}^*(B)$; Soare's theorem left the hope that the converse is true. Also, since the r.e. degrees of maximal sets are precisely the high r.e. degrees ($\boldsymbol{a}$ is high if $\boldsymbol{a}' = \boldsymbol{0}''$), Soare's theorem suggested connections

[†] During preparation of this paper, the first author was supported by the Heisenberg Programm der Deutschen Forschungsgemeinschaft, West Germany. The second author was partially supported by NSF Grant MSC 77-04013. The third author was partially supported by NSF Grant MSC 80-02937.

Received June 29, 1980 and in revised form December 20, 1980

between properties of r.e. sets invariant under automorphisms and the classification of r.e. degrees in terms of the jump operator. Another connection is Soare's theorem that if A is low ($A' \equiv_T \emptyset'$) then $\mathcal{L}^*(A) \cong \mathcal{E}^*$ [11, theorem 1.1].

Things are not so simple however. Lerman, Shore and Soare [4] gave examples of r.e. sets A and B such that $\mathcal{L}^*(A) \cong \mathcal{L}^*(B)$ and A is not automorphic to B; for $\mathcal{L}^*(A)$ they used the countable atomless Boolean algebra. Lerman and Soare [3] introduced a class $\mathcal{D}$ of r.e. sets invariant under automorphisms, the d-simple sets, such that the degrees of sets in $\mathcal{D}$ included some but not all low degrees ($\boldsymbol{a}$ is low if $\boldsymbol{a}' = \boldsymbol{0}'$). This result precluded a classification of the degrees of all invariant properties in terms of the jump operator as was hoped for by Martin, Shoenfield [7], and others.

The major accomplishment of this paper is to give examples of properties of r.e. sets which are invariant under Aut($\mathcal{E}$) but admit no classification in terms of the jump hierarchy of r.e. degrees or in terms of $\mathcal{L}^*(A)$. Specifically, we produce invariant classes of degrees which nontrivially split each class in the jump hierarchy of r.e. degrees. In particular, we refute a conjecture of Soare [10, conjecture 4.3] which says that every invariant class of degrees contains all the high degrees. We also show that for most known lattices of the form $\mathcal{L}^*(A)$, $\mathcal{L}^*(A) \cong \mathcal{L}^*(B)$ cannot imply that A is automorphic to B.

The outline of this paper is as follows. In §1 we study the promptly simple sets of Maass. The property of prompt simplicity is not invariant under automorphisms but has been used to construct automorphisms. We show that the promptly simple sets form a filter in $\mathcal{E}$ and the degrees of these sets form a filter in the r.e. degrees. In addition, the promptly simple sets give the first example of a property of r.e. sets, invariant or not, which splits the high degrees. In §2 we introduce the splitting property, a property which is invariant under Aut($\mathcal{E}$). We show how prompt simplicity led us to discover the splitting property. We show that the sets with the splitting property form a filter in $\mathcal{E}^*$ properly contained in the d-simple sets of Lerman and Soare mentioned above. We conclude that there are low d-simple sets which are not automorphic, answering a question of Lerman and Soare. In §3 we show that

$$\{B : B \text{ has the splitting property but is not hyperhypersimple}\}$$

is an invariant class of r.e. sets which nontrivially splits all the jump classes. We derive corollaries which show the ineffectiveness of degree or isomorphism type of $\mathcal{L}^*(A)$ for predicting the automorphism type of $\mathcal{E}$. In §4 we conclude with some questions.

We use the standard notation of Rogers [6]. Also $A =^* B$ denotes that the symmetric difference of A and B is finite. All sets and degrees are r.e.; $N = \{0, 1, 2, \cdots\}$. We use $\mathrm{dg}(A)$ for the Turing degree of A and $\leqq_T$, $\equiv_T$ for the relations between sets of Turing reducibility and equivalence. The letters $\boldsymbol{a}, \boldsymbol{b}, \cdots$ denote degrees. Let $\boldsymbol{H}_n = \{\boldsymbol{a} : \boldsymbol{a}^{(n)} = \boldsymbol{0}^{(n+1)}\}$ and $L_n = \{\boldsymbol{a} : \boldsymbol{a}^{(n)} = \boldsymbol{0}^{(n)}\}$. If $\{U_n\}_{n \in N}$ is a recursive sequence of r.e. sets, a simultaneous enumeration of $\{U_n\}_{n \in N}$ is a recursive function g with range $\{\langle m, n \rangle \mid m \in U_n\}$. (Here $\langle x, y \rangle$ is some effective pairing function mapping $N \times N$ one-one onto N.) If g is fixed, $U_{n,s} = \{x : (\exists t \leqq s)\ [g(t) = \langle x, n \rangle]\}$, (intuitively x is enumerated in U_n by the end of stage s). Define

$$U_n \backslash U_m = \{x : (\exists s)[x \in U_{n,s} - U_{m,s}]\} \quad \text{and} \quad U_n \searrow U_m = (U_n \backslash U_m) \cap U_m.$$

An enumeration of a single r.e. set A, is a recursive sequence of finite sets $\{A_s\}_{s \in N}$ such that $A_s \subseteq A_{s+1}$ for all s. Thus we allow finitely many elements to be enumerated in A at stage s. We identify sets with their characteristic functions; $A[x]$ is the restriction of A to arguments $\leqq x$. $\Phi_e(A)$ is the eth Turing reduction from oracle A. $\{W_e\}_{e \in N}$ is the canonical listing of the r.e. sets.

§1. Promptly simple sets

An r.e. set A is promptly simple if it is simple and the witnesses to the simplicity of A are enumerated in A "promptly."

DEFINITION 1.1. A coinfinite r.e. set A is *promptly simple* if there is a nondecreasing recursive function p and an enumeration $\{A_s\}_{s \in N}$ of A so that for every $e \in N$

$$W_e \text{ infinite} \Rightarrow (\exists s)(\exists x)[x \in (W_{e,s} - W_{e,s-1}) \cap A_{p(s)}]. \tag{1.1}$$

For example, the usual construction of a simple set produces a promptly simple set with p the identity function. (Although this definition of prompt simplicity is enumeration dependent, Theorem 1.3(ii) gives an equivalent definition which shows that prompt simplicity is independent of the enumeration of A.)

Maass introduced the promptly simple sets in [5] in connection with his work on a notion of genericity for r.e. sets. There he proved

THEOREM 1.2. (Maass, [5, theorem 17]). *If A and B are promptly simple and $\bar{A}$ and $\bar{B}$ are semilow ($\{e : W_e \cap \bar{A} \neq \varnothing\} \leqq_T \varnothing'$ and similarly for $\bar{B}$) then there is an automorphism Φ of $\mathscr{E}$ such that $\Phi(A) = B$.*

While Theorem 1.2 gives a large class, $\mathscr{PS} = \{A : A$ is promptly simple and semilow} of r.e. sets which are automorphic, it seems unlikely that the class $\mathscr{PS}$ forms an orbit (i.e., $A \in \mathscr{PS}$ and B automorphic to A implies $B \in \mathscr{PS}$).

Before establishing the properties of prompt simplicity mentioned in the introduction, we give some equivalent definitions of prompt simplicity which are especially useful in constructions.

THEOREM 1.3. *The following are equivalent:*

(i) *A is promptly simple.*

(ii) *A is coinfinite and there is a recursive function f such that for all $e \in N$*

$$W_{f(e)} \subseteq W_e, \tag{1.2}$$

$$W_{f(e)} \cap \bar{A} = W_e \cap \bar{A}, \quad \textit{and} \tag{1.3}$$

$$W_e \textit{ infinite} \Rightarrow W_e - W_{f(e)} \neq \emptyset. \tag{1.4}$$

(iii) *The same as* (ii) *but with* (1.4) *replaced by*

$$W_e \textit{ infinite} \Rightarrow W_e - W_{f(e)} \textit{ infinite}. \tag{1.5}$$

(If f is a function as in (iii), *we say f witnesses that A is promptly simple.)*

PROOF. (i) $\Rightarrow$ (ii). Given an enumeration $\{A_s\}_{s \in N}$ of A and a recursive function p which satisfy (1.1), let

$$W_{f(e)} = \{x : (\exists s)[x \in W_{e,s} - A_{p(s)}]\}.$$

$W_{f(e)}$ is r.e. uniformly in e and $W_{f(e)}$ certainly satisfies (1.2) and (1.3). By (1.1), if W_e is infinite, $W_e - W_{f(e)} \neq \emptyset$.

(ii) $\Rightarrow$ (iii). Let h be a recursive function so that $W_{h(e,x)} = W_e - \{0, 1, \cdots, x-1\}$. If W_e is infinite, so is $W_{h(e,x)}$ for every x. Let f be as in (ii). Let f' be defined by

$$W_{f'(e)} = \left\{ x : x \in \bigcap_{y \leqq x} \{W_{f(h(e,y))}\} \right\}.$$

Certainly $W_{f(h(e,y))} \subseteq W_e$ so that $W_{f'(e)} \subseteq W_e$. $W_{f'(e)} \cap \bar{A} = W_e \cap \bar{A}$ since if $x \in W_e \cap \bar{A}$, $x \in W_{h(e,y)} \cap \bar{A}$ for each $y \leqq x$. But $W_e - W_{f'(e)}$ is infinite whenever W_e is infinite since $W_{h(e,y)} - W_{f(h(e,y))}$ is nonempty whenever W_e is infinite; i.e. $W_e - W_{f'(e)}$ has an element greater than y for any fixed y.

(iii) $\Rightarrow$ (i). Given an enumeration $\{A_s\}_{s \in N}$ of A and a function f as in (iii), we define p to satisfy (1.1). Let

$$p(s) = (\mu t)(\forall x)(\forall e)[x \in W_{e,s} \Rightarrow x \in A_t \cup W_{f(e),t}];$$

p is recursive since for each s there are only finitely many pairs x, e such that $x \in W_{e,s}$ and for each of these pairs there is a $t \geqq s$ so that $x \in A_t \cup W_{f(e),t}$ by (1.3). That p satisfies (1.1) is a direct consequence of (1.5) and the definition of p.

□

Like most other classes of simple sets, the promptly simple sets form a filter in $\mathscr{E}$.

THEOREM 1.4. *The promptly simple sets together with the cofinite sets form a filter in* $\mathscr{E}$.

PROOF. If $A \subseteq A'$ and f witnesses that A is promptly simple, f witnesses that A' is promptly simple.

Suppose that f^A witnesses that A is promptly simple and f^B witnesses that B is promptly simple. We show that $A \cap B$ is promptly simple by a witness f which we construct below.

Assume that we have a simultaneous enumeration of A, B, and $\{W_e\}_{e \in N}$. Let g be a recursive function so that for all e, $W_{g(e)} = W_e \cap (A \setminus W_{f^A(e)})$. Define

$$W_{f(e)} = (W_{f^A(e)} \setminus A) \cup (W_{f^B(g(e))} \setminus B).$$

$W_{f(e)} \subseteq W_e$ since $W_{f^A(e)} \subseteq W_e$ and $W_{f^B(g(e))} \subseteq W_{g(e)} \subseteq W_e$. Thus (1.2) is satisfied for f. For (1.3), suppose that $x \in W_e \cap \overline{(A \cap B)}$. We need to show that $x \in W_{f(e)}$. If $x \in \bar{A}$, then $x \in W_{f^A(e)}$ and so $x \in W_{f(e)}$. If $x \in A \cap \bar{B}$ then either $x \in W_{f^A(e)} \setminus A$ (and thus $x \in W_{f(e)}$) or else $x \in W_{g(e)}$ so that $x \in W_{f^B(g(e))}$ and hence $x \in W_{f(e)}$. For (1.5), suppose that W_e is infinite. Then $W_{g(e)} = W_e \cap (A \setminus W_{f^A(e)})$ is infinite since $W_e - W_{f^A(e)}$ is infinite. But then $W_{g(e)} - W_{f^B(g(e))}$ is infinite. Each member of $W_{g(e)} - W_{f^B(g(e))}$ must be in $W_e - W_{f(e)}$. □

We next study the degrees of promptly simple sets. Our main conclusions are that this class of degrees is a filter in the upper semilattice of r.e. degrees and that this filter splits every jump class ($\boldsymbol{H}_n$ or $\boldsymbol{L}_n$) nontrivially. (F is a *filter* in an upper semilattice if (i) $\boldsymbol{a} \in F$ and $\boldsymbol{a} \leqq \boldsymbol{b}$ implies $\boldsymbol{b} \in F$ and (ii) $\boldsymbol{a}, \boldsymbol{b} \in F$ and $\boldsymbol{a} \cap \boldsymbol{b}$ exists implies $\boldsymbol{a} \cap \boldsymbol{b} \in F$.)

THEOREM 1.5. *Let A and B be promptly simple sets. Then there is a promptly simple set $C \leqq_T A, B$, $C' \equiv_T \varnothing'$.*

COROLLARY 1.6. *The degrees of promptly simple sets, $\mathscr{P}$, form a filter in the upper semilattice of r.e. degrees.*

PROOF OF COROLLARY 1.6. We first show that if $\boldsymbol{a} \in \mathscr{P}$ and $\boldsymbol{b} \geqq \boldsymbol{a}$ then $\boldsymbol{b} \in \mathscr{P}$. Let A be a promptly simple set, $\mathrm{dg}(A) = \boldsymbol{a}$. Let $C \leqq_T A$ be promptly simple so that if $c = \mathrm{dg}(C)$, $c' = 0'$. By a theorem of Lachlan [2, theorem 1], C has supersets of every degree $b \geqq c$ and so of every degree $b \geqq a$. But each such superset is promptly simple by Theorem 1.4.

Suppose that $a, b \in \mathcal{P}$ and $d = a \cap b$ exists. Then $(\exists c)\,[c \in \mathcal{P}\ \&\ c \leqq a, b]$. But then $c \leqq d$ so that $d \in \mathcal{P}$. □

Proof of Theorem 1.5. We assume that $A \cup B$ is coinfinite. We explain how to omit this assumption at the end of the proof. By Theorem 1.4, $A \cap B$ is promptly simple, say by witness f. To construct C promptly simple, it suffices to make C coinfinite and meet the following requirement for each $e \in N$:

$$P_e : W_e \text{ infinite} \Rightarrow (\exists x)(\exists s)\,[x \in (W_{e,s+1} - W_{e,s}) \cap C_{s+1}].$$

(Then C is promptly simple by (1.1) with $p(s) = s$.)

To make C of low r.e. degree it suffices [9, theorem 4.1] to meet the following requirement for every $e \in N$:

$$N_e : \text{If } \Phi_{e,s}(C_s; e) \text{ is defined for infinitely many } s \text{ then } \Phi_e(C; e) \text{ is defined.}$$

Let $r(e, s)$ be the length of the initial segment of C_s being used in the computation $\Phi_{e,s}(C_s; e)$ if it is defined and 0 otherwise. During the construction, to aid in making $C \leqq_T A, B$ we will enumerate certain r.e. sets U_e uniformly in e. By the recursion theorem, we may suppose we know the index of a recursive function g so that $U_e = W_{g(e)}$ for all $e \in N$.

Stage $s+1$. Find the least e such that P_e is not satisfied and

$$(\exists x)\,[x \in W_{e,s+1} - W_{e,s},\ x \geqq 3e, \text{ and } (\forall i \leqq e)\,[x > r(i, s)]].$$

(The clause $x \geqq 3e$ is used to make C coinfinite and also to remove the assumption $A \cup B$ coinfinite.) For the greatest such x enumerate into $U_e = W_{g(e)}$ all of $\{y : y \leqq x \text{ and } y \in \overline{A_s \cup B_s}\}$. Now find the least $t \geqq s$ such that

(i) $W_{f(g(e)),t} = U_{e,s+1}$, or

(ii) $(\exists y \leqq x)[y \in (A_t \cap B_t) - (A_s \cup B_s)]$.

In this latter case, enumerate x in C_{s+1}. Otherwise proceed to stage $s+2$. In either case say that P_e receives attention at stage $s+1$. (To see that either (i) or (ii) must happen note that whenever $y \in U_{e,s+1} - U_{e,s}$, $y \in (\overline{A_s \cup B_s}) \cap W_{g(e)}$. Thus either y must later be enumerated in $W_{f(g(e))}$ or into $A \cap B$; these two cases are reflected in (i) and (ii) respectively.)

Lemma 1.7. $C \leqq_T A$, $C \leqq_T B$.

Proof. If $A[x] = A_t[x]$ then $C[x] = C_t[x]$ because if $x \in C_{s+1} - C_s$, then $(\exists y \leqq x)\ (\exists t \geqq s)\ [y \in A_{t+1} - A_t]$. Similarly for B.

LEMMA 1.8. *Each requirement P_e receives attention only finitely often.*

PROOF. If P_e receives attention infinitely often, $U_e = W_{g(e)}$ is infinite. Thus $(\exists y)[y \in U_e - W_{f(g(e))}]$. If y was enumerated in U_e at stage $s+1$ then P_e received attention at stage $s+1$ and case (i) could not happen because of y. Thus (ii) happens, and so P_e is satisfied at stage $s+1$ and never again receives attention.

LEMMA 1.9. *Each requirement N_e is satisfied.*

PROOF. This is the usual finite injury argument; the clause $(\forall i \geqq e)$ $[x > r(i,s)]$ guarantees that if s_0 is a stage such that no requirement P_i, $i < e$ receives attention after s_0 then $\Phi_e(A;e)$ is defined iff $\Phi_{e,s}(A_s;e)$ is defined for *some* $s \geqq s_0$.

LEMMA 1.10. *Each requirement P_e is satisfied.*

PROOF. If W_e is infinite, there are infinitely many x and stages s such that $s \geqq$ last stage at which all P_i, $i < e$ receive attention and $x \in W_{e,s+1} - W_{e,s}$, $x \geqq 3e$, and $x \geqq \lim_s r(i,s)$ for each $i \leqq e$ (which all exist by 1.9). Thus P_e will receive attention enough times until it is met by Lemma 1.8.

Note that the clause $x \geqq 3e$ guarantees that C is coinfinite, Lemma 1.9 that C is low, and Lemma 1.10 that C is promptly simple. To remove the assumption that $A \cup B$ is coinfinite, first apply the proof using A for both A and B to get $\hat{A} \leqq_T A$ such that A is low, promptly simple, and $|\hat{A}[3e]| \leqq e$. Similarly, get $\hat{B} \leqq_T B$, low, promptly simple, and $|\hat{B}[3e]| \leqq e$. Now apply the proof using $\hat{A}$ and $\hat{B}$ for A and B, $\hat{A} \cup \hat{B}$ is coinfinite. □

Theorem 1.5 implies that no pair of promptly simple sets form a minimal pair (A, B are a minimal pair if dg(A) and dg(B) have infimum **0** in the r.e. degrees). In fact, no promptly simple set is half of a minimal pair.

THEOREM 1.11. *If A is promptly simple then A is not half of a minimal pair.*

PROOF. The proof is much like that of Theorem 1.5 so we just give a sketch. Given a nonrecursive set B, we must find an r.e. set C so that $C \leqq_T A$, $C \leqq_T B$, and C is nonrecursive. We have the usual simplicity requirements to make C nonrecursive:

$$P_e: \qquad W_e \text{ infinite} \Rightarrow C \cap W_e \neq \emptyset.$$

Again, to meet P_e whle insuring that $C \leqq_T A$ we will enumerate certain r.e. sets U_e and assume by the recursion theorem that $U_e = W_{g(e)}$ for some fixed recursive function g. To insure that $C \leqq_T B$ we will require that if $B[x] = B_s[x]$

then $C[x] = C_s[x]$. Assume that for each s, $B_{s+1} - B_s$ has exactly one element; denote this element by b_{s+1}. Let A be promptly simple with witness f.

CONSTRUCTION. Stage $s + 1$. Find the least e such that

$$W_{e,s+1} \cap C_s = \emptyset \qquad \text{and}$$

$$(\exists x)\,[x \in W_{e,s+1}, x \geqq 2e, \text{and } x > b_{s+1}].$$

For the greatest such x enumerate into $U_e = W_{g(e)}$ all of $\{y : y \leqq x \text{ and } y \in \bar{A}_s\}$. Find the least $t \geqq s$ such that

(i) $W_{f(g(e)),t} = U_{e,s+1}$ or

(ii) $(\exists y \leqq x)\ [y \in A_t - A_s]$.

In the latter case, enumerate x into C_{s+1}.

Virtually the same argument as Lemma 1.8 establishes that requirement P_e receives attention only finitely often. Each requirement P_e is satisfied since if W_e is infinite there are infinitely many stages s so that

$$(\exists x)\,[x \in W_{e,s+1}, x > 2e, \text{and } x > b_{s+1}],$$

else B is recursive. □

COROLLARY 1.12. *$\mathcal{P}$ splits $\boldsymbol{H}_1$ nontrivially.*

PROOF. $\mathcal{P} \cap \boldsymbol{H}_1 \neq \emptyset$ by Corollary 1.6. Lachlan has shown [1, theorem 2] that there are high r.e. degrees $\boldsymbol{a}, \boldsymbol{b}$ such that $\boldsymbol{a} \cap \boldsymbol{b} = \boldsymbol{0}$. At least one of $\boldsymbol{a}, \boldsymbol{b}$ is not in $\mathcal{P}$ so $\mathcal{P} \supsetneqq \boldsymbol{H}_1$. □

COROLLARY 1.13. *There are maximal sets A and B such that A is promptly simple but B is not. Thus, since A is automorphic to B, prompt simplicity is not invariant under automorphisms of $\mathcal{E}$.*

PROOF. For A take a maximal superset of a low promptly simple set. (Every low r.e. set has a maximal superset.) For B take any maximal set in some high degree $\boldsymbol{h} \notin \mathcal{P}$. Every high degree contains a maximal set. □

§2. The splitting property

Although prompt simplicity is not definable in the lattice $\mathcal{E}^*$, it led us to the discovery of the following property which is.

DEFINITION 2.1. A has the *splitting property* if for every r.e. set B there are r.e. sets B_0 and B_1 so that

(2.1) $$B_0 \cup B_1 = B,$$

(2.2) $$B_0 \cap B_1 = \emptyset,$$

(2.3) $$B_0 \subseteq A,$$

(2.4) If W is r.e. but $W - B$ is not r.e. then $W - B_0$ and $W - B_1$ are not r.e.

We say B_0 and B_1 are a *splitting* of B over A if (2.1)–(2.4) hold.

Friedberg showed that N has the splitting property; a splitting of B over N is called a Friedberg splitting of B.

THEOREM 2.2. *If A is promptly simple then A has the splitting property.*

PROOF. Let A be promptly simple with witness f. Given B r.e., we show how to enumerate B_0 and B_1 to split B over A. We have the following requirements for each pair $\langle e, i\rangle$, $e \in N$, $i = 0, 1$:

$$P_{\langle e,i\rangle}: \qquad B_i \neq \bar{W}_e.$$

The method of meeting these requirements will guarantee 2.4. $P_{\langle e,i\rangle}$ is *satisfied* at stage s if $B_{i,s} \cap W_{e,s} \neq \emptyset$.

Let g be a recursive function such that $W_{g(e)} = W_e \searrow B$.

Stage $s+1$. We assume $|B_{s+1} - B_s| = 1$. Let $x \in B_{s+1} - B_s$. Let $\langle e, i\rangle$ be the least pair such that $P_{\langle e,i\rangle}$ is not satisfied at stage s and $x \in W_{e,s}$. If $i = 1$ or $\langle e, i\rangle$ does not exist, enumerate x in $B_{1,s+1}$. Otherwise, ($i = 0$), let $t \geqq s$ be a stage such that $x \in A_t \cup W_{f(g(e)),t}$. Such a stage exists since $x \in W_{g(e)}$. Then enumerate $x \in B_{0,s+1}$ if $x \in A_t$, otherwise enumerate $x \in B_{1,s+1}$. We say $\langle e, i\rangle$ *receives attention* at stage $s+1$.

Note that the construction guarantees (2.1), (2.2) and (2.3).

LEMMA 2.3. *For each pair $\langle e, i\rangle$, $P_{\langle e,i\rangle}$ receives attention at only finitely many stages.*

PROOF. By induction suppose that $P_{\langle e',i'\rangle}$ never receives attention after stage s_0 if $\langle e', i'\rangle < \langle e, i\rangle$.

If $i = 1$ the lemma is obvious since if $P_{\langle e,i\rangle}$ receives attention at stage $s+1$, $P_{\langle e,i\rangle}$ is satisfied at all stages $t \geqq s+1$. Suppose then that $i = 0$ and $P_{\langle e,i\rangle}$ receives attention infinitely often.

If $\langle e, 0\rangle$ receives attention at stage $s+1$ then $(\exists x)\ [x \in W_{e,s} \cap (B_{s+1} - B_s)]$. Any such x is in $W_{g(e)}$ so that $W_{g(e)}$ is infinite. But then suppose that $s+1 \geqq s_0$ is any stage so that $(\exists x)[x \in W_{e,s} \cap (B_{s+1} - B_s)]$ but $x \not\in W_{f(g(e))}$. Then $\langle e, 0\rangle$ receives attention at stage $s+1$ and is satisfied at stage $s+1$. This is a contradiction since $\langle e, 0\rangle$ will never again receive attention.

LEMMA 2.4. *If $W_e \searrow B$ is infinite then $W_e \cap B_i \neq \emptyset$.*

PROOF. The proof of the last lemma really showed that if $W_e \searrow B$ is infinite, $P_{\langle e,i\rangle}$ receives attention enough times so that it is satisfied.

LEMMA 2.5. *If W is r.e. and $W - B$ is not r.e. then $W - B_i$ is not r.e. for $i = 0, 1$.*

PROOF. If $W - B_i = W_e$, then $W_e \searrow B$ is finite by Lemma 2.4. But then $W - B =^* W_e \setminus B$ so that $W - B$ is r.e. for a contradiction. □

Notice that indices for the sets B_0 and B_1 in the above proof can be produced uniformly from the index of B. In fact this stronger *uniform splitting property* is easily seen to be equivalent to prompt simplicity.

Like the promptly simple sets, the sets with the splitting property form a filter in $\mathscr{E}$.

THEOREM 2.6. *$\mathscr{S}_{\mathscr{P}} = \{A : A$ has the splitting property$\}$ is a filter in $\mathscr{E}$.*

PROOF. $\mathscr{S}_{\mathscr{P}}$ is closed upwards for if $A \subseteq A'$ and B_0 and B_1 split B over A, B_0 and B_1 split B over A'.

Suppose that A and C have the splitting property and B is an r.e. set. We show that B splits over $A \cap C$. Let B_0 and B_1 split B over A and B_{00} and B_{01} split B_0 over C. We claim that B_{00} and $B_{01} \cup B_1$ split B over $A \cap C$. Certainly $B_{00} \cup (B_{01} \cup B_1) = B$ and $B_{00} \cap (B_{01} \cup B_1) = \emptyset$. $B_{00} \subseteq A \cap C$ since $B_0 \subseteq A$ and $B_{00} \subseteq C, B_0$. Suppose that $W - B$ is not r.e. Then $W - B_0$ is not r.e. so that $W - B_{00}$ is not r.e. Now $W - (B_{01} \cup B_1)$ is not r.e., since otherwise $W - B_1 = (W - (B_{01} \cup B_1)) \cup (B_{01} \cap W)$ would be r.e. □

We next show how the splitting property is related to other properties of simplicity. (Note that if $A \in \mathscr{S}_{\mathscr{P}}$ and A is coinfinite then A is indeed simple.) In particular we show that all such r.e. sets A are d-simple. The d-simple sets were introduced by Lerman and Soare [3] and gave the first example of a lattice definable class of sets whose r.e. degrees nontrivially split a jump class. (There are low d-simple sets, but there are low r.e. degrees which contain no d-simple set.) In §3 we will show that $\{A : A$ is not hyperhypersimple and $A \in \mathscr{S}_{\mathscr{P}}\}$ is a class of r.e. sets whose degrees nontrivially split every jump class of r.e. degrees.

THEOREM 2.7. (i) *If A is hyperhypersimple then A has the splitting property.*

(ii) *If A has the splitting property and is coinfinite then A is d-simple.*

PROOF. (i) Suppose to the contrary that A does not have the splitting property and B is a set which does not split over A. Let B_0 and B_1 be a Friedberg splitting of B; then $B_0 \cap \bar{A}$ and $B_1 \cap \bar{A}$ must be nonempty else B_0 and B_1 split B

over A. Let B_{00} and B_{01} be a Friedberg splitting of B_0. $B_{00} \cap \bar{A}$ ($B_{01} \cap \bar{A}$) is nonempty else B_{00} and $B_{01} \cup B_1$ (B_{01} and $B_{00} \cup B_1$) are a splitting of B over A as in the proof of Theorem 2.6. Continue to get sets $B_1, B_{01}, B_{001}, \cdots$ which are disjoint r.e. sets each intersecting A. This is a weak array witnessing that A is not hyperhypersimple.

(ii) A is *d-simple* if

$$(\forall X)(\exists Y)(\forall Z)[X \cap \bar{A} = Y \cap \bar{A}, Y \subseteq X, \text{ and } Z - X \text{ infinite} \Rightarrow (Z - Y) \cap A \text{ infinite}]. \tag{2.5}$$

Suppose that A has the splitting property. Given X, let B_0 and B_1 split X over A. Then $Y = B_1$ is the desired Y in the definition of d-simple. Certainly $Y \subseteq X$ and $Y \cap \bar{A} = X \cap \bar{A}$. Suppose that $Z - X$ is infinite. If $Z - X$ is r.e., $(Z - X) \cap A$ is nonempty by the simplicity of A. If $Z - X$ is not r.e. then $Z - B_0$ is not r.e. so in particular $Z \cap B_0$ is infinite. But $Z \cap B_0 \subseteq (Z - Y) \cap A$. □

The next theorem answers a question of Lerman and Soare. They asked whether any two low d-simple sets are automorphic.

THEOREM 2.8. *There are d-simple sets A and C of low r.e. degree such that A has the splitting property but C does not.*

PROOF. For A take any low promptly simple set (see Theorem 1.5). Lerman and Soare [3, theorem 3.1] have constructed r.e. sets C and D such that C and D are d-simple but $C \cap D$ is not d-simple. Their construction can easily be combined with requirements to make C and D low. But C and D cannot both have the splitting property else $C \cap D$ does and so is d-simple.

It is also possible to construct C directly using a technique for constructing sets without the splitting property that imposes negative restraint similar to that of the construction of a minimal pair. □

It is still an open problem to give invariants which characterize the orbit of some low r.e. set. Theorem 2.8 says that d-simplicity (and in fact uniform d-simplicity as can be seen from the proof) is not enough; it also suggests the problem may be very hard.

§3. Degrees of sets with the splitting property

THEOREM 3.1. *Suppose that A has the splitting property, A is coinfinite, and A is not hyperhypersimple. Then A is not half of a minimal pair.*

PROOF. We first need a lemma, the proof of which is essentially an idea of Lachlan [2, theorem 1].

LEMMA 3.2. *If A is not hyperhypersimple, and C is any r.e. set, there is an r.e. set B such that*

(i) $A \cap B \leqq_T A$,

(ii) $B \leqq_T C$,

(iii) *if B is recursive, $C \leqq_T A$.*

PROOF. Let f be a function witnessing the nonhyperhypersimplicity of A; i.e.,

$$(\forall e)[W_{f(e)} \cap \bar{A} \neq \emptyset],$$

$$(\forall e)[W_{f(e)} \text{ is finite}],$$

$$(\forall e)(\forall i)[e \neq i \Rightarrow W_{f(e)} \cap W_{f(i)} = \emptyset].$$

Also we will assume that $W_{f(e)}$ has no member $\leqq e$. Define B from a simultaneous enumeration of A, C, and $\{W_e\}_{e \in N}$ as follows:

$$B = \{x : (\exists e)(\exists s)[x \in (W_{f(e),s} - A_s) \text{ and } e \in (C_{s+1} - C_s)]\}.$$

B is in Σ_1 form and so is r.e. (Informally, enumerate x in B if e appears in C while $x \in W_{f(e)} - A$.)

(i) $A \cap B \leqq_T A$ as follows: If $x \in A$ find s so that $x \in A_s$. Then

$$x \in B \leftrightarrow (\exists e < x)(\exists t < s)[x \in (W_{f(e),t} - A_t) \text{ and } e \in (C_{t+1} - C_t)].$$

This last condition can be checked recursively. (Note that if $x \in W_{f(e)}$ then $e < x$.)

(ii) $B \leqq_T C$ since if s is a stage that $C_s[x] = C[x]$, $x \in B$ iff $(\exists e < x)(\exists t < s)$ $[x \in (W_{f(e),t} - A_t)$ and $e \in (C_{t+1} - C_t)]$.

(iii) Suppose that B is recursive. To see if $e \in C$, find an element $x \in \bar{A}$ such that $x \in W_{f(e)}$ $(W_{f(e)} \cap \bar{A} \neq \emptyset)$. Find an s such that $x \in W_{f(e),s}$. Then $e \in C$ iff $e \in C_s$ or $x \in B$. Thus $C \leqq_T A$.

LEMMA 3.3. *If C is r.e., nonrecursive, then C and A do not form a minimal pair.*

PROOF. We may suppose that $C \nleqq_T A$. Let B_0, B_1 split B over A where B is the set constructed above. B is nonrecursive by Lemma 3.2 (iii). Then B_0 and B_1 are nonrecursive. (Take W in (2.4) to be N.) We claim that $B_0 \leqq_T A, C$.

$B_0 \leqq_T C$ since $B \leqq_T C$ and $B_0 \leqq_T B$. (To see if $x \in B_0$, see if $x \in B$. If $x \in B$ wait until $x \in B_0$ or in B_1.)

$B_0 \leqq_T A$. We show how to decide if $x \in B_0$. First see if $x \in B \cap A$.

($B \cap A \leqq_T A$ by Lemma 3.2(i).) If $x \notin B \cap A$, then $x \notin B_0$. If $x \in B \cap A$, wait until $x \in B_0$ or in B_1 and answer appropriately. $\square$

Notice that we did not use all of the splitting property in Theorem 3.1.

DEFINITION 3.4. An r.e. set A has the *weak splitting property* if for every r.e. B there are sets B_0, B_1 such that

$$B_0 \cup B_1 = B,$$

$$B_0 \cap B_1 = \emptyset,$$

$$B_0 \subseteq A,$$

$$B \text{ nonrecursive} \Rightarrow B_0, B_1 \text{ are nonrecursive.}$$

Of course the weak splitting property is on its face weaker than the splitting property; we have only replaced (2.4) by a weaker condition.

COROLLARY 3.5. *Suppose that A has the weak splitting property, A is coinfinite, and A is not hyperhypersimple. Then A is not half of a minimal pair.*

We now derive some corollaries and discuss the significance of this result. If $\mathscr{C}$ is any class of r.e. sets, let $\boldsymbol{\mathscr{C}} = \{\boldsymbol{a} : (\exists A)\ [\mathrm{dg}(A) = \boldsymbol{a} \text{ and } A \in \mathscr{C}]\}$. Let $\mathscr{C}^* = \{A : \neg(\exists B \supseteq A)\ [B \in \mathscr{C}]\}$. Let $\mathscr{H} = \{A : A \text{ is hyperhypersimple}\}$. Let $\mathscr{S}_{\mathscr{P}} = \{A : A \text{ has the splitting property}\}$.

COROLLARY 3.6.(i) $\boldsymbol{\mathscr{S}_{\mathscr{P}}} \supseteq \boldsymbol{H}_1$ *but* $\boldsymbol{\mathscr{S}_{\mathscr{P}}}$ *nontrivially splits* $\boldsymbol{H}_{n+1}$ *and* $\boldsymbol{L}_n$ *for every* $n \geqq 1$.

(ii) $\boldsymbol{\mathscr{S}_{\mathscr{P}}} - \boldsymbol{\mathscr{H}}$ *nontrivially splits* $\boldsymbol{H}_n$ *and* $\boldsymbol{L}_n$ *for every* $n \geqq 1$.

(iii) $\mathscr{S}_{\mathscr{P}} \cap \mathscr{H}^*$ *is a filter in* $\mathscr{E}$; $\boldsymbol{\mathscr{S}_{\mathscr{P}} \cap \mathscr{H}^*}$ *splits* $\boldsymbol{H}_1$.

PROOF. There is a high r.e. degree $\boldsymbol{a}$ such that $\boldsymbol{a}$ is half of a minimal pair [1, theorem 2]. Thus

$$\mathrm{dg}(B) \leqq \boldsymbol{a} \Rightarrow [B \in \mathscr{S}_{\mathscr{P}} \Leftrightarrow B \in \mathscr{H}].$$

(i) $\boldsymbol{H}_{n+1} \not\subseteq \boldsymbol{\mathscr{S}_{\mathscr{P}}}$: Choose $\boldsymbol{b} < \boldsymbol{a}$, $\boldsymbol{b} \in \boldsymbol{H}_{n+1}$. $\boldsymbol{b} \notin \boldsymbol{\mathscr{S}_{\mathscr{P}}}$ since $\boldsymbol{b} \notin \boldsymbol{\mathscr{H}}$ ($\boldsymbol{\mathscr{H}} = \boldsymbol{H}_1$). (Similarly $\boldsymbol{L}_n \not\subseteq \boldsymbol{\mathscr{S}_{\mathscr{P}}}$.)

$\boldsymbol{L}_n \cap \boldsymbol{\mathscr{S}_{\mathscr{P}}} \neq \emptyset$, $\boldsymbol{H}_{n+1} \cap \boldsymbol{\mathscr{S}_{\mathscr{P}}} \neq \emptyset$: Let $\boldsymbol{c}$ be low such that $\boldsymbol{c}$ contains a promptly simple set. Then choose $\boldsymbol{b} \geqq \boldsymbol{c}$, $\boldsymbol{b} \in \boldsymbol{H}_{n+1}(\boldsymbol{L}_n)$. $\boldsymbol{b} \in \boldsymbol{\mathscr{S}_{\mathscr{P}}}$ since $\boldsymbol{b}$ contains a promptly simple set by Theorem 1.6.

$\boldsymbol{H}_1 \subseteq \boldsymbol{\mathscr{S}_{\mathscr{P}}}$: as $\boldsymbol{\mathscr{S}_{\mathscr{P}}} \supseteq \boldsymbol{\mathscr{H}} = \boldsymbol{H}_1$.

(ii) Since $\boldsymbol{\mathscr{H}} = \boldsymbol{H}_1$, $\boldsymbol{\mathscr{S}_{\mathscr{P}}} - \boldsymbol{\mathscr{H}}$ and $\boldsymbol{\mathscr{S}_{\mathscr{P}}}$ coincide on $\overline{\boldsymbol{H}_1}$. Thus we need only show that $\mathscr{S}_{\mathscr{P}} - \mathscr{H}$ splits $\boldsymbol{H}_1$. Note that $\boldsymbol{a} \notin \mathscr{S}_{\mathscr{P}} - \mathscr{H}$. To find $\boldsymbol{b} \in \mathscr{S}_{\mathscr{P}} - \mathscr{H}$, $\boldsymbol{b} \in \boldsymbol{H}_1$, we choose a promptly simple set B such that $\mathscr{L}^*(B)$ is infinite but no element of $\mathscr{L}^*(B)$ other than 0 or 1 is complemented. (Such a set is called r-maximal and satisfies $\mathrm{dg}(B) \in \boldsymbol{H}_1$.)

Note that the standard constructions of r-maximal sets are easily combined with requirements to make the sets promptly simple. One can also see that there is such a promptly simple set B as follows: if A is r.e. there is a promptly simple set B such that $\mathscr{L}^*(B) \cong \mathscr{L}^*(A)$. This follows from the fact that there is a promptly simple set $\hat{B}$ such that $\mathscr{L}^*(\hat{B}) \cong \mathscr{E}^*$ (any low r.e. promptly simple set will do) and any superset of B is promptly simple.

(iii) We need only check that $(\mathscr{S}_{\mathscr{P}} \cap \mathscr{H}^{\#}) \cap \boldsymbol{H}_1 \neq \varnothing$. However, the set B constructed in (ii) above is in $\mathscr{S}_{\mathscr{P}} \cap \mathscr{H}^{\#}$. $\mathscr{S}_{\mathscr{P}} \cap \mathscr{H}^{\#}$ is obviously a filter in $\mathscr{E}$. □

Call $\mathscr{C}$ invariant if $\mathscr{C}$ is a class of r.e. sets invariant under $\mathrm{Aut}(\mathscr{E})$. Soare conjectured [9, conjecture 4.3] that every invariant class contains $\boldsymbol{H}_1$. Corollary 3.6 (iii) shows this false, even if we restrict it to definable filters in $\mathscr{E}$. Corollary 3.6(ii) gives an example of an invariant class which splits each $\boldsymbol{H}_n$ and $\boldsymbol{L}_n$; such classes had previously been found only for $\boldsymbol{L}_1$.

If B is r.e. there is an $A \in \mathscr{S}_{\mathscr{P}}$ such that $\mathscr{L}^*(B) \cong \mathscr{L}^*(A)$. This follows from the fact that there is a $C \in \mathscr{S}_{\mathscr{P}}$ such that $\mathscr{L}^*(C) \cong \mathscr{E}^*$ and supersets of C are in $\mathscr{S}_{\mathscr{P}}$.

Thus if $\mathrm{dg}(B) \leqq \boldsymbol{a}$, $\boldsymbol{a}$ as in the proof of Corollary 3.6, $\mathscr{L}^*(B) \cong \mathscr{L}^*(A)$ cannot imply B automorphic to A unless B is hyperhypersimple. However for most known lattices of the form $\mathscr{L}^*(B)$, $(\exists A)\,[\mathrm{dg}(A) = \boldsymbol{a}$ and $\mathscr{L}^*(A) \cong \mathscr{L}^*(B)]$. Thus it seems unlikely that $\mathscr{L}^*(A) \cong \mathscr{L}^*(B)$ implies A automorphic to B in any case where A is not hyperhypersimple. Corollary 3.6 also implies that no set in $\mathscr{S}_{\mathscr{P}} - \mathscr{H}$ can be taken by an automorphism to any set recursive in $\boldsymbol{a}$. This is the first example of a high degree which omits some nontrivial automorphism type and gives an answer to problem number 60 of H. Friedman [0].

§4. Open questions

What invariant properties of low r.e. sets A and B guarantee that A and B are automorphic? It seems unlikely that the splitting property is enough although this is still open.

Are there any lattices $\mathscr{L}$ other than finite lattices such that $\mathscr{L} \cong \mathscr{L}^*(A) \cong \mathscr{L}^*(B)$ implies A automorphic to B? We doubt it; by our remarks at the end of the last section, the only reasonable candidates are Boolean algebras. Even there, Lerman, Shore and Soare have ruled out the countable atomless Boolean algebra.

Are any classes $\boldsymbol{H}_n$ or $\bar{\boldsymbol{L}}_n$ invariant other than $\boldsymbol{H}_1$, $\bar{\boldsymbol{L}}_0$, and $\bar{\boldsymbol{L}}_2$?

Are there any invariant classes which are definable in the language of r.e. degrees; 0, <, $\cup$, 1? In particular, if $\boldsymbol{a}$ is not half of a minimal pair is there a B, $\mathrm{dg}(B) = \boldsymbol{a}$ such that B is not hyperhypersimple but has the weak splitting property?

Added in proof. Shore has recently shown that the degrees which are not halves of minimal pairs are precisely the degrees of promptly simple sets and so has answered this question.

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